J. Appl. Maths Mechs Vol. 55, No. 5, pp. 724–732, 1991
Printed in Great Britain.

0021-8928/91 $15.00+.00

THE RAY METHOD FOR INVESTIGATING TRANSIENT WAVE PROCESSES IN A THIN ELASTIC ANISOTROPIC LAYER†

YU. A. ROSSIKHIN and M. V. SHITIKOVA

Voronezh

(*Received* 6 *April* 1990)

The ray method is used to solve the boundary-value problems that lead to the propagation of two-dimensional shock waves in anisotropic plates of constant thickness, taking into account rotatory inertia and transverse shear deformations, as well as the coupling of extensional and transverse vibrational modes. The essence of the method is to construct the solution beyond the shock wave fronts by using ray series similar to Taylor series. Transient two-dimensional wave propagation in semi-infinite AT-cut quartz plates is studied.

INTRODUCTION

TRANSIENT wave propagation in an anisotropic plate was the main topic discussed in [1]. The dynamical behaviour of elastic anisotropic plates taking into account rotatory inertia and transverse shear deformations was considered in [2–4]. In particular, differential equations were derived in [2] for the coupled extensional and transverse vibrational modes of an anisotropic plate of constant thickness. These take the form of two subsystems: the equations of a generalized plane strained state and equations of the Timoshenko type. However, to solve problems of the vibrations of such plates, one introduces simplifying assumptions which uncouple the two subsystems.

1. THE RAY METHOD

The stress-strain state of a thin AT-cut quartz plate, taking into account transverse shear deformations and rotatory inertia, as well as the coupling of extensional and transverse vibrational modes, is described by a system of equations which may be found in [2].

Let us assume that some dynamical excitation applied at the boundary of the plate $y = x_1\nu_1 + x_3\nu_3 = 0$, where $\nu = \cos\varphi$, $\nu_3 = \sin\varphi$ are the components of the normal vector to the boundary, induces the formation of strong surfaces of discontinuity in the plate, each a cylindrical surface $S(t)$ with directrix $L(t)$ in the plane of the plate x_1, x_3 and generators parallel to the x_2 axis. Beyond the surface $S(t)$ the desired functions $Z(x_1, x_3, t)$ are expanded in ray series [5]

†*Prikl. Mat. Mekh.* Vol. 55, No. 5, pp. 849–856, 1991.

$$Z(x_1, x_3, t) = \sum_{k=0}^{\infty} \frac{1}{k!} [Z_{,(k)}] |_{t=yG^{-1}} \left(t - \frac{y}{G}\right)^k H\left(t - \frac{y}{G}\right) \tag{1.1}$$

where $[Z_{,(k)}]$ are the jumps of the kth derivative of t with respect to time $Z(x_1, x_2, t)$ across the shock wave front, G the normal velocity of the wave, y is the distance measured from the plate boundary along the normal to the boundary and $H(t)$ is the Heaviside unit step function.

To determine the coefficients of the ray series (1.1), we differentiate the transport equations of the system (see [2]) k times with respect to t, differentiate the relationship between the force factors and displacements $k+1$ times, and evaluate their difference across the wave surface $S(t)$. The result is

$$\begin{gathered}
[N_{1,1(k)}] + [N_{5,3(k)}] = 2b\rho\, [v_{1,(k+1)}], \ [N_{5,1(k)}] + [N_{3,3(k)}] = 2b\rho\, [v_{3,(k+1)}] \\
[Q_{1,1(k)}] + [Q_{3,3(k)}] = 2b\rho\, [v_{2,(k+1)}] \\
[M_{1,1(k)}] + [M_{5,3(k)}] - [Q_{1,(k)}] = {}^2/{}_3 b^3\rho\, [\Phi_{1,(k+1)}] \\
[M_{5,1(k)}] + [M_{3,3(k)}] - [Q_{3,(k)}] = {}^2/{}_3 b^3\rho\, [\Phi_{3,(k+1)}] \\
[N_{\alpha,(k+1)}] = 2b\, \{\bar{c}_{1\alpha}\, [v_{1,1(k)}] + \bar{c}_{\alpha 3}\, [v_{3,3(k)}] + K_{\alpha/2+5/2} \bar{c}_{\alpha 4}\, ([v_{2,3(k)}] + \\
+ [\Phi_{3,(k)}])\} \\
[N_{5,(k+1)}] = 2b\, \{c_{55}\, ([v_{3,1(k)}] + [v_{1,3(k)}]) + K_1 c_{56}\, ([v_{2,1(k)}] + [\Phi_{1,(k)}])\} \\
[Q_{1,(k+1)}] = 2b\, \{K_1 c_{56}\, ([v_{3,1(k)}] + [v_{1,3(k)}]) + K_1^2 c_{66}\, ([v_{2,1(k)}] + [\Phi_{1,(k)}])\} \\
[Q_{3,(k+1)}] = 2b\, \{K_3 \bar{c}_{14}\, [v_{1,1(k)}] + K_3 \bar{c}_{34}\, [v_{3,3(k)}] + K_3^2 \bar{c}_{44}\, ([v_{2,3(k)}] + \\
+ [\Phi_{3,(k)}])\} \\
[M_{\alpha,(k+1)}] = {}^2/{}_3 b^3\, (\gamma_{1\alpha}\, [\Phi_{1,1(k)}] + \gamma_{\alpha 3}\, [\Phi_{3,3(k)}]), \ \alpha = 1, 3 \\
[M_{5,(k+1)}] = {}^2/{}_3 b^3 \gamma_{55}\, ([\Phi_{3,1(k)}] + [\Phi_{1,3(k)}]) \\
K_1^2 = \frac{\pi^2}{12}, \quad K_3^2 = \frac{\pi^2}{24\bar{c}_{44}} \{c_{22} + c_{44} - [(c_{22} - c_{44})^2 + 4c_{24}^2]^{1/2}\} \\
\bar{c}_{pq} = c_{pq} - \frac{c_{2p} c_{q2}}{c_{22}}, \quad \gamma_{pq} = \bar{c}_{pq} - \frac{\bar{c}_{4p} \bar{c}_{q4}}{\bar{c}_{44}}, \quad \gamma_{55} = c_{55} - \frac{c_{56}^2}{c_{66}}
\end{gathered} \tag{1.2}$$

Here N_1, N_2 and N_3 are the forces acting in the plane of the plate, Q_1 and Q_3 are the shear forces, M_1 and M_3 are the bending moments, M_5 is the twisting moment, $2b$ is the plate thickness, ρ is the density, v_1, v_2 and v_3 are the displacement velocities Φ_1 and Φ_3 are the rotatory angular velocities of the normal to the middle surface of the plate and c_{pq} are the moduli of a quartz crystal; an index following a comma indicates differentiation with respect to the appropriate variable.

In the sequel we shall use the summation convention with respect to repeated indices not in parentheses; unless otherwise stated, Latin indices take the values 1, 2, 3 and Grek indices take the values 1, 2.

Using the compatibility relations for discontinuities of the kth derivative of $Z(x_1, x_2, t)$ [6]

$$G[Z_{,\alpha(k)}] = -[Z_{,(k+1)}]\, \nu_\alpha + \frac{d[Z_{,(k)}]}{dt}\, \nu_\alpha + G[Z_{,(k)}]_{,s} \tau_\alpha \quad (\alpha = 1, 3)$$

we transform Eqs (1.2), to obtain

$$
\begin{aligned}
&\rho\left(G_{(n)}^{2}-G^{2}\right) X_{(k+1)}^{(n)}=2 \rho G_{(n)}^{2} \frac{d X_{(k)}^{(n)}}{d t}+G p_{i j} l_{i}^{(n)} \sum_{f=1}^{3} X_{(k), s}^{(f)} l_{j}^{(f)}+ \\
&\quad+G d_{i \alpha} l_{i}^{(n)} \sum_{\alpha=1}^{2} Y_{(k)}^{(\gamma)} l_{\alpha}^{(\gamma *)}-F_{i(k-1)} l_{i}^{(n)} \\
&\rho\left(G_{(\gamma *)}^{2}-G^{2}\right) Y_{(k+1)}^{(\gamma)}=2 \rho G_{(\gamma *)}^{2} \frac{d Y_{(k)}^{(\gamma)}}{d t}+G p_{\alpha \beta}^{*} l_{\beta}^{(\gamma *)} \sum_{\delta=1}^{2} Y_{(k), s}^{(\delta)} l_{\alpha}^{(\delta *)}+ \\
&\quad+G d_{\alpha i}^{*} l_{\alpha}^{(\gamma *)} \sum_{f=1}^{3} X_{(k)}^{(f)} l_{i}^{(f)}-U_{\alpha(k-1)} l_{\alpha}^{(\gamma *)} \\
&F_{i(k-1)} l_{i}^{(n)}=\rho G_{(n)}^{2} \frac{d^{2} X_{(k-1)}^{(n)}}{d t^{2}}+p_{i j} l_{i}^{(n)} G \sum_{f=1}^{3} \frac{d X_{(k-1), s}^{(f)}}{d t} l_{j}^{(f)}+ \\
&\quad+G^{2} \bar{s}_{i j} l_{i}^{(n)} \sum_{f=1}^{3} X_{(k-1), s s}^{(f)} l_{j}^{(f)}+d_{i \alpha} l_{i}^{(n)} G \sum_{j=1}^{2} \frac{d Y_{(k-1)}^{(\gamma)}}{d t} l_{\alpha}^{(\gamma *)}+ \\
&\quad+G^{2} \bar{c}_{i \alpha} l_{i}^{(n)} \sum_{\gamma=1}^{2} Y_{(k-1), s}^{(\gamma)} l_{\alpha}^{(\gamma *)}, \quad U_{\alpha(k-1)} l_{\alpha}^{(\gamma *)}=\rho G_{(\gamma *)}^{2} \frac{d^{2} Y_{(k-1)}^{(\gamma)}}{d t^{2}}+ \\
&\quad+p_{\alpha \beta}^{*} l_{\alpha}^{(\gamma *)} G \sum_{\delta=1}^{2} \frac{d Y_{(k-1), s}^{(\delta)}}{d t} l_{\beta}^{(\delta *)}+\bar{s}_{\alpha \beta}^{*} l_{\alpha}^{(\gamma *)} G^{2} \sum_{\delta=1}^{2} Y_{(k-1), s s}^{(\delta)} l_{\beta}^{(\delta *)}+ \\
&\quad+d_{\alpha i}^{*} l_{\alpha}^{(\gamma *)} G \sum_{f=1}^{3} \frac{d X_{(k-2)}^{(f)}}{d t} l_{i}^{(f)}+\bar{d}_{\alpha i}^{*} l_{\alpha}^{(\gamma *)} G^{2} \sum_{f=1}^{3} X_{(k-1), s}^{(f)} l_{i}^{(f)}
\end{aligned}
\tag{1.3}
$$

$$
\begin{gathered}
X_{(k)}^{(f)}=X_{i(k)} l_{i}^{(f)}, \quad X_{1(k)}=\left[v_{1,(k)}\right], \quad X_{2(k)}=\left[v_{3,(k)}\right], \quad X_{3(k)}=\left[v_{2,(k)}\right] \\
Y_{(k)}^{(\gamma)}=Y_{\alpha(k)} l_{\alpha}^{(\gamma *)}, \quad Y_{1(k)}=\left[\Phi_{1,(k)}\right], \quad Y_{2(k)}=\left[\Phi_{3,(k)}\right] \\
p_{11}=2\left(\bar{c}_{11} \tau_{1} \nu_{1}+c_{55} \tau_{3} \nu_{3}\right), \quad p_{12}=p_{21}=\left(\bar{c}_{13}+c_{55}\right)\left(\tau_{1} \nu_{3}+\tau_{3} \nu_{1}\right) \\
p_{13}=p_{31}=\left(K_{3} \bar{c}_{14}+K_{1} c_{56}\right)\left(\tau_{1} \nu_{3}+\tau_{3} \nu_{1}\right), \quad p_{22}=2\left(\bar{c}_{33} \tau_{3} \nu_{3}+c_{55} \tau_{1} \nu_{1}\right) \\
p_{23}=p_{32}=2\left(K_{1} c_{56} \tau_{1} \nu_{1}+K_{3} \bar{c}_{34} \tau_{3} \nu_{3}\right), \quad p_{33}=2\left(K_{1}{ }^{2} c_{66} \tau_{1} \nu_{1}+K_{3}{ }^{2} \bar{c}_{44} \tau_{3} \nu_{3}\right) \\
d_{11}=K_{1} c_{56} \nu_{3}, \quad d_{21}=K_{1} c_{56} \nu_{1}, \quad d_{31}=K_{1}{ }^{2} c_{66} \nu_{1}, \quad d_{12}=K_{3} \bar{c}_{14} \nu_{1} \\
d_{22}=K_{3} \bar{c}_{34} \nu_{3}, \quad d_{32}=K_{3}{ }^{2} \bar{c}_{44} \nu_{3}, \quad d_{11}{ }^{*}=-3 b^{-2} d_{11} \nu_{1} \nu_{3}{ }^{-1}, \quad d_{12}{ }^{*}=-3 b^{-2} d_{21} \nu_{1}{ }^{-1} \nu_{3}
\end{gathered}
$$

The quantities $q_{\alpha\beta}{}^*$ are derived from $p_{\alpha\beta}$ by replacing $\bar{c}_{mn}(m, n = 1, 3)$ in the latter by γ_{mn} and c_{55} by γ_{55}, $1 = x_2\tau_1 + x_3\tau_3\, (\tau_1 = -\sin\varphi,\ \tau_3 = \cos\varphi)$ is the distance measured from the origin along the wave surface, $\rho G^2_{(f)}$, $\rho G^2_{(\gamma^*)}$ are the principal values and $l_i^{(f)}$, $l_\alpha^{(\gamma^*)}$ the unit principal directions of the symmetric tensors $s_{ij} = \frac{1}{2}p_{ij}|_{\tau_\delta=\nu_\delta}$, $s^*_{\alpha\beta} = \frac{1}{2}p^*_{\alpha\beta}|_{\tau_\delta=\nu_\delta}$ $(\delta = 1, 3)$, respectively; the tensors $\bar{s}_{ij}$, $\bar{s}^*_{\alpha\beta}$ are obtained from s_{ij}, $s^*_{\alpha\beta}$ by replacing ν_δ in the latter by τ_δ $(\delta = 1, 2)$; finally, $d_{13}{}^*$, $d_{23}{}^*$, $d_{22}{}^*$, $d_{21}{}^*$ are derived from the corresponding asterisk-free quantities by multiplying the latter by $-3b^{-2}$.

Confining our attention from now on to three terms of the radial series (1.1) for the unknown functions, we put $k = 0, 1, 2$ in Eqs (1.3) to obtain

$$
\begin{gathered}
X_{(0)}^{(f,f)} = h_{(0)}^{(f)}(z),\ X_{(0)}^{(n,f)} = 0\ (n \neq f),\ X_{(0)}^{(\alpha*,f)} = 0 \\
Y_{(0)}^{(\alpha*,\alpha)} = h_{(\alpha)}^{(\alpha*)}(z),\ Y_{(0)}^{(\alpha*,\beta)} = 0\ (\alpha * \neq \beta),\ Y_{(0)}^{(n,\alpha)} = 0 \\
X_{(1)}^{(f,f)} = h_{(1)}^{(f)}(z) + tH_{(0)}^{(f,f)}, \quad X_{(1)}^{(n,f)} = a_{(f,n)}^{(n)} h_{(0),z}^{(n)} \quad (n \neq f) \\
X_{(1)}^{(\alpha*,f)} = a_{(f,\alpha*)}^{(\alpha*)} h_{(0)}^{(\alpha*)}, \quad Y_{(1)}^{(\alpha*,\alpha)} = h_{(1)}^{(\alpha*)} + tM_{(0)}^{(\alpha*,\alpha)} \\
Y_{(1)}^{(\alpha*,\beta)} = d_{(\beta,\alpha*)}^{(\alpha*)} h_{(0),z}^{(\alpha*)} \quad (\alpha \neq \beta), \quad Y_{(1)}^{(n,\alpha)} = d_{(\alpha,n)}^{(n)} h_{(0)}^{(n)} \\
X_{(2)}^{(f,f)} = h_{(2)}^{(f)}(z) + tH_{(1)}^{(f,f)} + {}^1/_2 t^2 \chi_{(1)}^{(f,f)}, \quad X_{(2)}^{(n,f)} = H_{(1)}^{(n,f)} + ta_{(f,n)}^{(n)} H_{(0),z}^{(n,n)} \quad (n \neq f) \\
X_{(2)}^{(\alpha*,f)} = H_{(1)}^{(\alpha*,f)} + ta_{(f,\alpha*)}^{(\alpha*)} M_{(0)}^{(\alpha*,\alpha)}, \quad Y_{(2)}^{(\alpha*,\alpha)} = h_{(2)}^{(\alpha*)}(z) + tM_{(1)}^{(\alpha*,\alpha)} + {}^1/_2 t^2 \chi_{(1)}^{(\alpha*,\alpha)} \\
Y_{(2)}^{(\alpha*,\beta)} = M_{(1)}^{(\alpha*,\beta)} + td_{(\beta,\alpha*)}^{(\alpha*)} M_{(0),s}^{(\alpha*,\alpha)} \quad (\alpha \neq \beta), \quad Y_{(2)}^{(n,\alpha)} = M_{(1)}^{(n,\alpha)} + td_{(\alpha,n)}^{(n)} H_{(0)}^{(n,n)}
\end{gathered} \tag{1.4}
$$

Here the first superscript in parentheses indicates the mode number (the first three modes are numbered 1, 2, 3 and the fourth and fifth 1*, 2*), and the second indicates the specific projection; $h_{(k)}^{(1)}$, $h_{(k)}^{(2)}$, $h_{(k)}^{(3)}$ $(k = 0, 1, 2)$ are arbitrary functions of $z = 1 - g_{(f)}t$, and $h_{(k)}^{(1^*)}$, $h_{(k)}^{(2^*)}$ $(k = 0,\ 1,\ 2)$ are arbitrary functions of $z^* = 1 - g_{(\alpha^*)}t$ (for convenience, the indices of z have been omitted in the formulae); the functions $H_{(k)}^{(n,f)}$, $M_{(k)}^{(\alpha^*,\alpha)}$ $(k = 0.\ 1)$, $M_{(1)}^{(n,\alpha)}$, $\chi_{(1)}^{(f,f)}$ and $\chi_{(1)}^{(\alpha^*,\alpha)}$, which depend on the aforementioned arbitrary functions and their derivatives with respect to z, will not be given here, as their expressions are rather cumbersome

$$
g_{(n)} = \frac{p_{ij} l_i^{(n)} l_j^{(n)}}{2\rho G_{(n)}}, \quad g_{(\gamma*)} = \frac{p_{\alpha\beta}^{*} l_\alpha^{(\gamma*)} l_\beta^{(\gamma*)}}{2\rho G_{(\gamma*)}} \tag{1.5}
$$

Using (1.4) we can construct a solution in the form of a ray series (1.1) with a suitable Heaviside function, for each of the five shock-wave modes. As the problem is linear, the final result is obtained by simply adding together the ray series thus constructed. The five sets of arbitrary functions appearing in the solution are determined by five boundary conditions. We will consider several types of boundary conditions.

Let us assume first that three quantities are specified at the edge of the plate: the deformation velocity and two rotatory angular velocities of the normal to the middle surface of the plate, as functions of the time t and cordinate 1, i.e.

$$
v_n|_{y=0} = \sum_{k=0}^{\infty} X_{n(k)}^0(s)\frac{t^k}{k!}, \quad v_\tau|_{y=0} = \sum_{k=0}^{\infty} X_{\tau(k)}^0(s)\frac{t^k}{k!}
$$

$$
v_2|_{y=0} = \sum_{k=0}^{\infty} X_{3(k)}^0(s)\frac{t^k}{k!}, \quad \Phi_n|_{y=0} = \sum_{k=0}^{\infty} Y_{n(k)}^0\frac{t^k}{k!}
$$

$$
\Phi_\tau|_{y=0} = \sum_{k=0}^{\infty} Y_{\tau(k)}^0\frac{t^k}{k!}
$$

This yields the following relations for determining the arbitrary functions:

$$
\sum_{f=1}^{3} h_{(0)}^{(f)} C_N^{(f)} = X_{N(0)}^0, \quad \sum_{\alpha=1}^{2} h_{(0)}^{(\alpha*)} B_M^{(\alpha*)} = Y_{M(0)}^0
$$

$$
\sum_{f=1}^{3} h_{(1)}^{(f)} C_N^{(f)} = -\sum_{\substack{f=1\\(f\neq n)}}^{3}\sum_{n=1}^{3} a_{(f,n)}^{(n)} h_{(0),z}^{(n)} C_N^{(f)} - \sum_{f=1}^{3}\sum_{\alpha=1}^{2} a_{(f,\alpha*)}^{(\alpha*)} h_{(0)}^{(\alpha*)} C_N^{(f)} + X_{N(1)}^0
$$

$$
\begin{gathered}
\sum_{\alpha=1}^{2} h_{(1)}^{(\alpha*)} B_M^{(\alpha*)} = -\sum_{\substack{\alpha=1 \\ (\alpha\neq\beta)}}^{2} \sum_{\beta=1}^{2} d_{(\beta,\alpha*)}^{(\alpha*)} h_{(0),z}^{(\alpha*)} B_M^{(\beta*)} - \sum_{n=1}^{3}\sum_{\alpha=1}^{2} d_{(\alpha,n)}^{(n)} h_{(0)}^{(n)} B_M^{(\alpha*)} + Y_{M(1)}^{0} \\
\sum_{f=1}^{3} h_2^{(f)} C_N^{(f)} = -\sum_{\substack{f=1 \\ (f\neq n)}}^{3}\sum_{n=1}^{3} H_{(1)}^{(n,f)} C_N^{(f)} - \sum_{\alpha=1}^{2}\sum_{f=1}^{3} H_{(1)}^{(\alpha*,f)} C_N^{(f)} + X_{N(2)}^{0} \\
\sum_{\alpha=1}^{2} h_{(2)}^{(\alpha*)} B_M^{(\alpha*)} = -\sum_{\substack{\alpha=1 \\ (\alpha\neq\beta)}}^{2}\sum_{\beta=1}^{2} M_{(1)}^{(\alpha*,\beta)} B_M^{(\beta*)} - \sum_{n=1}^{3}\sum_{\alpha=1}^{2} M_{(1)}^{(n,\alpha)} B_M^{(\alpha*)} + Y_{M(2)}^{0}
\end{gathered}
\tag{1.6}
$$

In each system of equations the index N will take the values n, τ, 3, and the index M, the values n, τ and moreover

$$
C_n^{(f)} = l_1^{(f)}\nu_1 + l_2^{(f)}\nu_3, \quad C_\tau^{(f)} = l_1^{(f)}\tau_1 + l_2^{(f)}\tau_3, \quad C_3^{(f)} = l_3^{(f)},
$$
$$
B_n^{(\alpha*)} = l_1^{(\alpha*)}\nu_1 + l_2^{(\alpha*)}\nu_3, \quad B_\tau^{(\alpha*)} = l_1^{(\alpha*)}\tau_1 + l_2^{(\alpha*)}\tau_3
$$

If we assume that the quantities specified at the plate edge are the forces N_n, $N_{n\tau}$ in the plane of the plate, the shear force Q_n, the bending and twisting moments M_n, $M_{n\tau}$, then we can derive a system of equations similar to (6) for the arbitrary functions. When that is done the right-hand sides of the equations will involve, instead of $X_{N(k)}^0$, $Y_{M(k)}^0$ $(k = 0, 1, 2)$, the functions $N_{N(k)}^0$, $M_{N(k)}^0$ defined on the boundary ($N_{n(k)}^0$, $N_{\tau(k)}^0$ are the coefficients of the Maclaurin series of the boundary forces in the pane of the plate, $N_{2(k)}^0$ are the same for the boundary shear force, and $M_{n(k)}^0$, $M_{\tau(k)}^0$ are the same for the boundary bending and twisting moments, respectively).

Boundary conditions of other types may be treated in an analogous fashion.

2. EXAMPLE

The above formulae will now be used to investigate the dependence on the angle φ of the propagation velocities of the strong discontinuity surfaces $G_{(n)}$, $G_{(\gamma^*)}$ and of the velocities $g_{(n)}$, $g_{(\gamma^*)}$ at which the perturbations propagate along the wave surfaces. Figures 1 and 2 illustrate these magnitudes, in nondimensional form, in polar coordinates (curves 1, 2, 3 in Fig. 1a represent the velocities $\bar{G}_{(1)}$, $\bar{G}_{(2)}$, $\bar{G}_{(3)}$ and curves 1, 2 in Fig. 1b, the velocities $\bar{G}_{(1^*)}$, $\bar{G}_{(2^*)}$). For reasons of symmetry Fig. 1 shows only the upper parts of the curves and Fig. 2 the right-hand parts; it should be understood that on reflection in the vertical axis the solid curves in

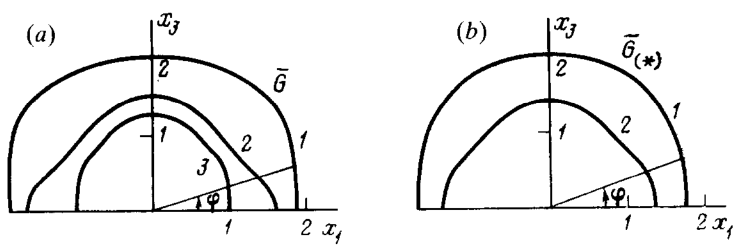

FIG. 1.

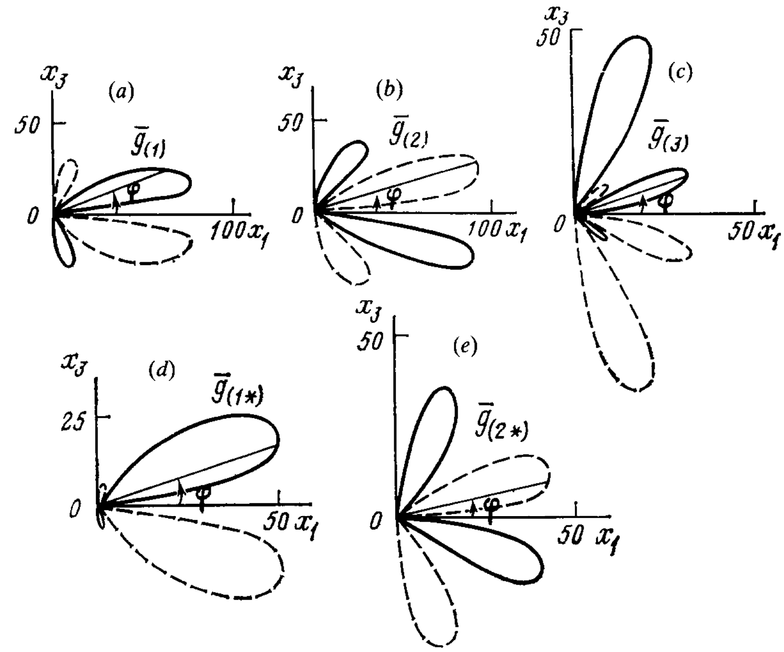

FIG. 2.

Fig. 2 become dashed curves and vice versa. All the velocities $G_{(i)}$, $G_{(\alpha^*)}$ are given in units of the least velocity along the direction of the X_1 axis and all velocities $g_{(i)}$, $g_{(\gamma^*)}$ are given in units of the least velocity in the direction of propagation $\varphi = 50°$.

It is clear from a comparison of the curves that the extremal values of the velocities $\bar{G}_{(n)}$, $\bar{G}_{(\gamma^*)}$ correspond to vanishing velocities $\bar{g}_{(n)}$, $\bar{g}_{(\gamma^*)}$; moreover, the quantities $\bar{g}_{(n)}$, $\bar{g}_{(\gamma^*)}$ change sign as they go through zero (the solid curves in Fig. 2a–e represent positive values and the dashed curves negative values), i.e. the radial tubes may deviate from the normal in either direction, depending on the direction in which the wave surface is propagating.

As examples of the solution of boundary-value problems, let us consider the propagation of plane waves in a quartz plate in the direction $\varphi = 0°$, triggered by a shock impulse of type (1.6), when all the quantities $X^0_{N(k)}$, $Y^0_{N(k)}$ except $X^0_{n(0)}$, $Y^0_{n(0)}$ vanish or when all the quantities $N^0_{N(k)}$, $M^0_{N(k)}$ except $N^0_{2(0)}$, $M^0_{\tau(0)}$ vanish.

A shock of the first type in the direction $\varphi = 0°$ gives rise to five wave modes, propagating at velocities $G_{(1)} > G_{(1^*)} > G_{(2)} > G_{(3)}$ (numbered in decreasing order of magnitude). In the first mode v_1 experiences a discontinuity, but Φ_3 is continuous together with its first derivative, and the second derivative of Φ_3 with respect to x_1 also has a jump. In the second mode Φ_1 experiences a discontinuity, as do the first derivative of v_1 and the second derivatives of v_2, v_3. In the third mode the first derivatives of Φ_1, v_2, v_3 and the second derivative of v_1 are discontinuous, in the fourth—the first derivatives of v_1 and Φ_3, and in the fifth—the first derivatives of Φ_1, v_2, v_3 and the second derivative of v_1. Numerical analysis of the solution with boundary conditions of the first type shows that the fourth mode does not make a significant contribution to determining the deformation velocities and its rotatory angular velocities may be ignored.

To illustrate, Fig. 3 shows the nondimensional deformation velocities and rotatory angular velocities (the former in units of the initial velocity X_0 and the latter in units of Y_0) for an A-cut quartz plate of thickness

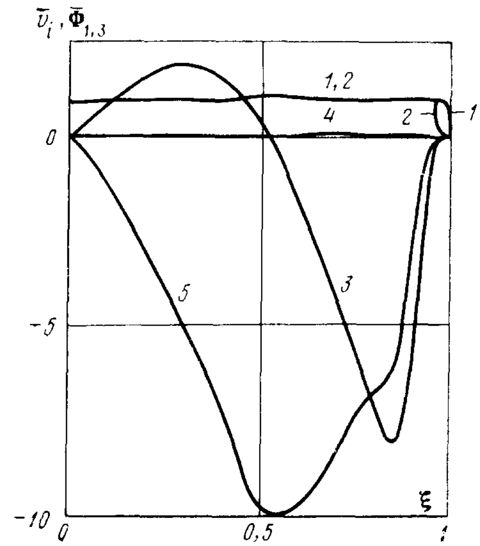

FIG. 3.

$2b = 2$ mm at $\tau = tY_0 = 1$, plotted against

$$\xi = \frac{x_1}{G(1)} Y_0$$

Curves 1–5 represent $\bar{v}_1$, Φ_1, $\bar{v}_3$, Φ_3, $\bar{v}_2$, respectively. Along the interval from 0 to 0.53 all five modes contribute to the solution; from 0.53 to 0.76—the first four, from 0.76 to 0.86—the first three, from 0.86 to 0.96—the first two, and from 0.96 to 1.00—the first wave only.

If the initial shock is of the second type, the nature of each of the five modes changes as follows. In the first mode the first derivative of v_1 and second derivatives of Φ_1 and Φ_3 experience discontinuities; in the second, the first derivative of Φ_1 and second derivatives of v_2 and v_3 are discontinuous, in the third—v_2 and v_3 and the first derivative of Φ_1, in the fourth—Φ_3, the first derivative of v_1 and the second derivative of Φ_1, and in the fifth—v_2 and v_3 and the first derivative of Φ_1. Analysis of computed data in this case imply that the first and second modes do not significantly affect the values of the deformation velocities and rotatory angular velocities and their contribution to the solution may be ignored. Curves of the nondimensional quantities are plotted in Fig. 4.

In conclusion, let us compute the ray velocities $G_{L(n)}$, $G_{L(\alpha^*)}$ of strong discontinuity modes (the velocities at which the perturbations travel along the ray), construct the ray velocity curves, compare them with the corresponding phase velocity curves $G_{(n)}$, $G_{(\alpha^*)}$ (Fig. 1a, b) and, in addition, determine the angles $\gamma_{(n)}$, $\gamma_{(\alpha^*)}$ that characterize the deviation of the rays from the wave normals. As already mentioned, the extrema of the phase velocities $G_{(n)}$, $G_{(\alpha^*)}$ are the zeros of the velocities $g_{(n)}$, $g_{(\alpha^*)}$ at which the perturbations travel along the appropriate wave fronts. This suggests that $g_{(n)}$, $g_{(\alpha^*)}$ are the derivatives with respect to φ of the phase velocities $G_{(n)}$, $G_{(\alpha^*)}$. Indeed, if we differentiate the expression

$$s_{ij} l_i^{(n)} l_j^{(n)} = \rho G_{(n)}^2$$

with respect to φ and use the fact that $s_{ij} l_i^{(n)} l_j^{(m)} = 0$ if $n \neq M$, this gives

$$G_{(n),\varphi} = s_{ij,\varphi} l_i^{(n)} l_j^{(n)} (2\rho G_{(n)})^{-1} \tag{2.1}$$

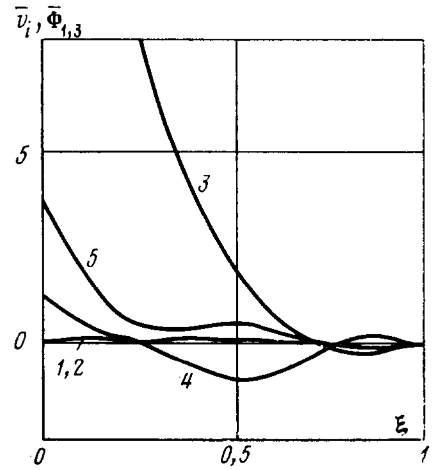

FIG. 4.

Since $s_{ij} = \frac{1}{2}p_{ij}|_{\tau_\sigma = \nu_\sigma}$ and so $s_{ij,\varphi} = p_{ij}$, it follows that formula (2.1) is just formula (1.5) for $g_{(n)}$, i.e. $G_{(n),\varphi} = g_{(n)}$. Similar reasoning yields $G_{(\alpha^*),\varphi} = g_{(\alpha^*)}$.

To determine the location of the perturbation at any fixed instant of time, we must eliminate the parameter φ from the system of equations

$$\begin{aligned} F\,(x_1,\, x_3,\, \varphi) &= x_1\nu_1 + x_3\nu_3 - Gt = 0 \\ F_1\,(x_1,\, x_3,\, \varphi) &= x_1\tau_1 + x_3\tau_3 - gt = 0 \end{aligned} \tag{2.2}$$

For convenience, we have omitted the index of the mode number.

Since $g = G_{,\varphi}$, we have $F_1(x_1, x_3, \varphi) = F_{,\varphi}(x_1, x_3, \varphi)$, i.e. the ray velocity curves are the envelopes of the wave fronts of plane waves radiating at a certain time from a point source placed at the origin [7].

Equations (2.2) yield the equations of the ray velocity surfaces in polar notation

$$\rho = \sqrt{x_1^2 + x_3^2} = G_L t, \quad G_L = \sqrt{G^2 + g^2} \tag{2.3}$$

and expressions for the angles

$$\cos\gamma = GG_L^{-1}, \quad \sin\gamma = gG_L^{-1} \tag{2.4}$$

describing the deviation of the rays from the wave normal.

Figure 5 shows the ray velocity curves for the five plane modes, determined by formula (2.2) (in view of symmetry only the upper parts of the curves are shown).

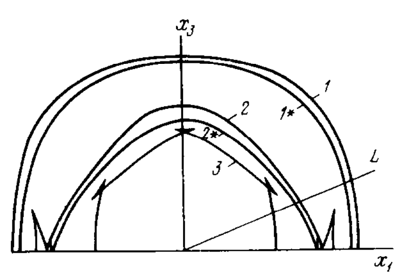

FIG. 5.

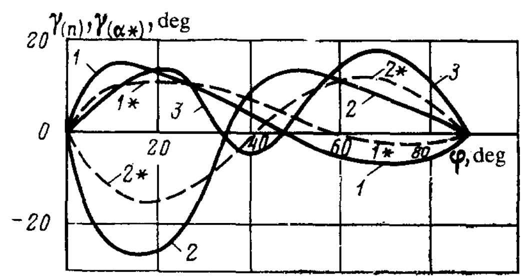

FIG. 6.

Obviously, smooth wave fronts are observed for only two modes—quasi-extensional (the first mode) and quasi-extensional rotatory (the second). In quasi-transverse (third and fifth) and quasi-transverse rotatory (fourth) modes the wave fronts contain lacunae: the third and fourth modes have two lacunae each, symmetrically located with respect to the x_1 axis, and the fifth has six lacunae, two of which lie on the x_3 axis and four are symmetrically located with respect to the origin, on straight lines at angles $+35° 15'$ to the x_1 axis (the lacunae are not symmetrical about the straight lines). We observe that the AT-cut is also inclined to the crystallographic axis at an angle of $35° 15'$. The ray L may intersect the wave surface at five, seven or nine points, i.e. along a ray in an AT-cut quartz plate up to nine elastic modes of different velocities may propagate, two of which are quasi-extensional and the others quasi-transverse.

In Fig. 6 the angles of deviation γ of the rays from the wave normal are plotted against the inclination φ of the wave normal to the x_1 axis, as obtained from (2.4).

A glance at the figure shows that all the curves except the third have two extrema and cut the φ axis at three points (the third curve has three extrema and four points of intersection with the φ axis). In other words, the rays deviate most from the normal in two directions (for the fifth mode—in three directions) and coincide with the wave normal in three directions (for the fifth mode—in four), two of which are the directions of the x_1, x_3 axes.

REFERENCES

1. COHEN H. and THOMAS R. S., Transient waves in inhomogeneous anisotropic elastic plates. *Acta Mech.* **58,** 12, 41–57, 1985.
2. MINDLIN R. D. and SPENCER W. J., Anharmonic, thickness-twist overtones of thickness-shear and flexural vibrations of rectangular, AT-cut quartz plates. *J. Acoust. Soc. Am.* **42,** 6, 1268–1277, 1967.
3. KACZKOWSKI Z., The influence of the shear forces and the rotatory inertia on the vibration of an anisotropic plate. *Arch. mech. stosowanej* **12,** 4, 531–552, 1960.
4. KRASYUKOV V. P., Vibrations of anisotropic plates taking into account rotatory inertia and shear deformation. *Nauchn. Trudy Saratov. Politekhn. Inst.* **23,** 107–110, 1966.
5. ACHENBACH J. D. and REDDY D. P., Note on wave propagation in linearly viscoelastic media. *Z. Angew. Mat. Phys.* **18,** 1, 141–144, 1967.
6. THOMAS T., *Plastic Flow and Fracture in Solids.* Academic Press, New York, 1961.
7. MUSGRAVE M. J. P., *Crystal Acoustics. Introduction to the Study of Elastic Waves and Vibrations in Crystals.* Holden-Day, San Francisco, 1970.

Translated by D.L.